Fantastic Quantum Theories and Where to Find Them

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We present a uniform compositional framework for the treatment of a large class of quantum-like theories. Specifically, we will be interested in theories of wavefunctions valued in commutative semirings, which give rise to some semiring-based notion of classical non-determinism via the Born rule (both in its familiar quadratic version and in its higher-order variants). The models obtained with our construction possess many of the familiar structures used in Categorical Quantum Mechanics and form the playground for a recent reconstruction result by Tull. We provide a bestiary of increasingly exotic examples: some well known, such as ordinary quantum theory, real quantum theory and relational quantum theory; some less known, such as hyperbolic quantum theory and p-adic quantum theory; and some entirely new, such as "parity quantum theory", "finite-field quantum theory" and "tropical quantum theory". The measurement scenarios arising within these theories can be studied using the sheaf-theoretic framework for non-locality and contextuality. Their computational complexity can similarly be studied within existing frameworks for affine and unitary circuits over commutative semirings. Finally, we discuss the structure of phases and its implications to non-locality and the Fourier transform.

1 Introduction

The construction of toy models plays a key role in many foundational efforts across mathematics, physics and computer science. In the foundations of quantum theory, toy models help to understand which abstract structural features of quantum systems—and their interface to the classical world—are involved in providing different kinds of non-classical behaviour. In turn, this informs practical research into quantum computation and communication technology, helping to cut down the noise and focus on those features that truly contribute to quantum advantage.

The categorical and diagrammatic methods from Categorical Quantum Mechanics (CQM) [4, 21, 27, 28, 45] have proven particularly well suited to the construction and study of toy models of quantum theory, with the majority of efforts focussed on Spekkens's toy model [9, 23, 69] and the more general *relational quantum theory*[1] [6, 22, 32, 33, 44, 53, 59, 76]. In the same years, toy models have been developed within a variety of other frameworks: examples include *real quantum theory*, of special interest in the context of generalised/operational probabilistic theories and the study of Jordan algebras [7, 10, 12, 17, 47, 74, 75], *hyperbolic quantum theory* [50, 51, 55], *p-adic quantum theory* [48, 49, 56, 57, 61, 73], and *modal quantum theory* [11, 63, 64].

When constructing a toy model, it is essential to consider both the quantum side and the corresponding quantum-classical interface: indeed, many such models result in notions of classical non-determinism which are different from the conventional probabilistic one, and special care needs to be taken in order to achieve a consistent treatment of classical systems. Examples of this phenomenon include the possibilistic non-determinism arising from Spekkens's toy model and relational quantum theory [1, 2, 5, 6, 23], the p-adic non-determinism arising from p-adic quantum theory [49], and the signed probabilities arising from

[1] Not to be confused with the *relational quantum mechanics* of Rovelli [60].

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hyperbolic quantum theory [2, 3]. In order to address this issue, we adopt the framework of Categorical Probabilistic Theories [35], which can simultaneously treat quantum-like systems and classical systems endowed with generic semiring-based notion of non-determinism. Categorical Probabilistic Theories have been introduced with the intent of bridging the gap between CQM and Operational Probabilistic Theories (OPTs) [16, 17, 18, 42, 43]: they aim to provide categorical and diagrammatic methods in the style of CQM to talk about the problems that OPTs are concerned with. As a side-product of their abstract categorical formulation, these theories natively admit a general, semiring-based notion of classical non-determinism, and are therefore perfect to construct and study exotic models[2].

In this work, we focus our attention on a very large class of finite-dimensional quantum-like theories, where wavefunctions of complex amplitude are replaced with wavefunctions valued in some arbitrary commutative semiring S equipped with the action of a finite abelian group G. In the quantum-classical transition, non-determinism is taken to arise via a generalisation of the Born rule obtained through the higher-order CPM construction [39]. As a consequence, classical non-determinism is naturally and necessarily modelled by the semiring R of *positive elements* for Sunder the action of G (generalising the traditional probabilistic semiring $\mathbb{R}^+$ of positive elements in $\mathbb{C}$ under the conjugating action of $\mathbb{Z}_2$). As an underlying model for S-valued wavefunctions we consider the category S-Mat, with objects in the form S^X for all finite sets X, and morphisms $S^X \to S^Y$ given by $S^{Y \times X}$, the S-semimodule of Y-by-X matrices with values in S (equipped with matrix composition and identities). The category S-Mat generalises fHilb $\simeq \mathbb{C}$-Mat, and is a dagger compact category with Kronecker product as tensor product. We model mixed-state quantum theory using the higher-order CPM construction [39] $\mathrm{CPM}_{\Phi,\Xi}(S\text{-Mat})$ and we recover the full quantum-classical theory (including, amongst other, super-selected systems) as the Karoubi envelope for $\mathrm{CPM}_{\Phi,\Xi}(S\text{-Mat})$—generalising the CP* construction [25, 26, 66]—which we show to be an R-probabilistic theory [35] for the semiring R of positive elements.

In Section 2 we recap the framework of categorical probabilistic theories, within which the classical aspects of our theories should be understood. In Section 3 we present our framework in full generality. In Sections 4 and 5, we show that our framework captures a number of well-studied quantum-like theories (plus a couple of entirely new ones). Many unique features of each theory have been studied in previous literature: in this work, we focus our attention on the structure of interference.

Interference is one of the key features of quantum theory and its study is of fundamental importance to physics and quantum computer science. The study of models displaying altered notions of interference is a necessary step on the way to understanding the physical and computational constraints of quantum theory and, perhaps some day, to overcoming them. Within our framework, we investigate two different ways in which quantum interference can be modified:

1. By maintaining the traditional quadratic Born rule, but changing the semiring that wavefunctions are valued into. This results in alternative phase groups, which we investigate from the point of view of computational advantage—using the Abelian Hidden Subgroup Problem as case study—and of non-locality—using Mermin-type arguments as case study. This is done in Section 4.

2. By using a higher-order Born rule, but keeping complex-valued wavefunctions. This results in probabilistic theories displaying higher-order interference, which previous work has shown to provide additional computational advantage. This is done in Section 5.

[2] Anything which is not probabilistic should be deemed to be more or less exotic, in the sense that conventional wisdom about classical systems might fail in one way or another. This includes all toy models mentioned above, except for real quantum theory.

2 Categorical Probabilistic Theories

The main intuition behind a generalised, semiring-based notion of classical non-deterministic systems is borrowed from computer science, where the use of (commutative) semirings to model resources used by automata is commonplace. We look at probabilities in physics as a resource modelling non-determinism of classical systems, with properties captured by those of the commutative semiring $\mathbb{R}^+$: from this perspective, it makes sense to study what classical non-determinism looks like when $\mathbb{R}^+$ is replaced by some other commutative semiring R. Interesting alternative choices for R which already appeared in the literature include the boolean semiring $\mathbb{B}$ and other locales (in relational quantum theory), the quasi-probabilistic semiring $\mathbb{R}$ (a field, in hyperbolic quantum theory), the p-adic semiring Q_p (another field, in p-adic quantum theory), and finite fields (in modal quantum theory).

One of the reason for the wide adoption of semirings in mathematics is that they capture the bare minimum algebraic structure required by matrix multiplication, with commutativity being a necessary addendum when a symmetric tensor product of matrices is of interest (as is the case in many physical applications). As our *category of classical R-probabilistic systems* we take the category R-Mat of free finite-dimensional R-semimodule and R-linear maps between them: objects are in the form R^X with X finite sets and morphisms $R^X \to R^Y$ form the free finite-dimensional R-semimodule of Y-by-X matrices with values in R. The category R-Mat is a compact closed symmetric monoidal category, with Kronecker product of matrices as tensor product. It is enriched in itself, and hence in commutative monoids (CMon-enriched), so that each homset $R^X \to R^Y$ comes with a *mixing operation* $+$ and an *impossible process* 0. The category R-Mat contains the category fSet of finite sets and functions (the category of *classical deterministic systems*) as a subcategory, and from fSet it inherits an environment structure $(\dashv_{R^X} : R^X \to R^{\mathbf{1}})_X$ [20,26] given by the *discarding maps* $\dashv_{R^X} := (p_x)_x \mapsto \sum_x p_x$.

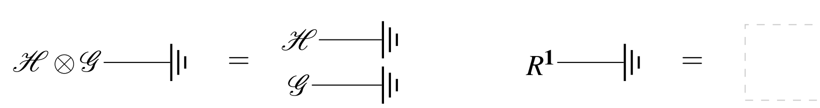
In Ref. [35], it is argued that the minimal requirements for a categorical probabilistic theory should include: (i) the explicit existence of classical systems[3]; (ii) the extendibility of probabilistic mixing to all systems[4]; (iii) the possibility of defining a meaningful notion of local state and discarding of systems[5].

Definition 2.1 (*R-probabilistic Theory*)**.**
An *R-probabilistic theory* is a symmetric monoidal category $\mathscr{C}$ which satisfies the following requirements.

(i) There is a full sub-SMC of $\mathscr{C}$, denoted by $\mathscr{C}_K$, which is equivalent to R-Mat.

(ii) The SMC $\mathscr{C}$ is enriched in commutative monoids, and the enrichment on the subcategory $\mathscr{C}_K$ coincides with the one given by the linear structure of R-Mat.

(iii) The SMC $\mathscr{C}$ comes with an environment structure, i.e. with a family $(\dashv_{\mathscr{H}} : \mathscr{H} \to \mathbf{1})_{\mathscr{H} \in \mathrm{obj}\,\mathscr{C}}$ of morphisms which satisfy the following requirements:



(2.1)

On the subcategory $\mathscr{C}_K$, this environment structure must coincide with the canonical one of R-Mat.

We refer to $\mathscr{C}_K$ as *classical theory*, and to its objects and morphisms as *classical systems* and *processes*. As diagrammatic convention, we use dashed wires for classical systems, and solid wires for generic ones.

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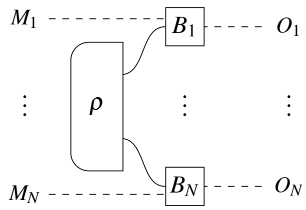
[3]So that the interface between classical and non-classical systems can be talked about in a compositional way. This includes, for example, classical control, classical outcomes, preparations and measurements.

[4]So that, for example, classical probabilistic control and marginalisation over classical outcomes are possible for all processes.

[5]Which are absolutely fundamental in most applications to quantum foundations and quantum protocols (but I acknowledge that Everettians and other faithful of the Church of the Larger Hilbert Space might disagree with me on this point).

R-probabilistic theories come with a number of native features that are commonplace in the modelling of quantum protocols: it is possible to exert classical control, to define tests with classical outcomes, to marginalise over classical outcomes, to work with preparations and measurements, and to apply any kind of classical pre- and post-processing. Amongst the many mixed quantum-classical processes, we can consider Bell-type measurement scenarios. An N-party *Bell-type measurement scenario* in an *R*-probabilistic theory is a process in the following form, where the processes $B_1,...,B_N$ and the state ρ are all normalised (recall that a process $f : \mathscr{H} \to \mathscr{K}$ is said to be *normalised* if $\dashv\!\Vdash_{\mathscr{K}} \circ f = \dashv\!\Vdash_{\mathscr{H}}$):



(2.2)

In the context of a Bell-type measurement scenario, the processes $B_1,...,B_N$ are often referred to as *measurements*, their inputs as *measurement choices* and their outputs as *(measurement) outcomes*. The following result from Ref. [35] ensures that non-locality in *R*-probabilistic theories can always be studied using the well-established sheaf-theoretic framework for non-locality and contextuality [2].

Theorem 2.2 (*Bell-type measurement scenarios [35]*)**.**
A Bell-type measurement scenario in an R-probabilistic theory always corresponds to a no-signalling empirical model in the sheaf-theoretic framework for non-locality and contextuality [2].

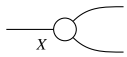
An immediate consequence of the connection with the sheaf-theoretic framework is that we can straightforwardly adapt a proof of Ref. [2] to rule out non-locality in a large class of toy models.

Theorem 2.3 (*Locality of R-probabilistic theories over fields*)**.**
If R is a field, then all R-probabilistic theories are local.

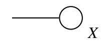
3 Quantum-like Theories

Note that two different linear structures intervene in the definition of quantum theory: the $\mathbb{C}$-linear structure of wavefunctions, modelling superposition, interference and phases, and the $\mathbb{R}^+$-linear structure of probability distributions over classical systems. We have already seen that the framework of *R*-probabilistic theories replaces the probability semiring $\mathbb{R}^+$ with a more general commutative semiring R as a model of classical non-determinism. In this Section, we construct a large class of toy models of quantum theory—which we refer to as *quantum-like theories*—by considering theories of wavefunctions with amplitudes valued in some commutative semiring S, generalising the field $\mathbb{C}$ traditionally used in quantum mechanics. To do so, we consider the symmetric monoidal category S-Mat, and we require classical non-determinism to arise via a generalisation of the Born rule, embodied by a higher-order CPM construction. The corresponding quantum-classical theory will therefore be modelled by (a full sub-SMC of) the Karoubi envelope for $\mathrm{CPM}_{\Phi,\Xi}(S\text{-Mat})$, where $\varphi : G \to \mathrm{Aut}(S)$ is a given action of a finite abelian group G on the semiring S (generalising the conjugation action of $\mathbb{Z}_2$ on $\mathbb{C}$), Φ is its extension by linearity to S-Mat and Ξ is a multi-environment structure, as defined by Ref. [39]. Our main result (Theorem 3.2) will show that this is an *R*-probabilistic theory, where R is the sub-semiring of positive elements of S (see definition below). In the remainder of this Section, we will assume that S, G, φ and Φ be fixed.

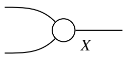
The category S-Mat is defined as in the previous Section. Each object S^X in S-Mat has at least one orthonormal basis $|x\rangle_{x\in X}$, as well as an associated special commutative Frobenius algebra $\circ_X$:



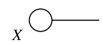
$$= \sum_{x\in X} |x\rangle \otimes |x\rangle \otimes \langle x|$$



$$= \sum_{x\in X} \langle x|$$



$$= \sum_{x\in X} |x\rangle \otimes \langle x| \otimes \langle x|$$

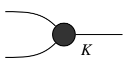


$$= \sum_{x\in X} |x\rangle \qquad (3.1)$$

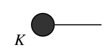
The category S-Mat is dagger compact, with matrix transpose as dagger: with respect to this dagger, the Frobenius algebras $\circ_X$ above and $\bullet_K$ below are always dagger Frobenius algebras. Cups and caps on all systems are given by the the special commutative †-Frobenius algebras $\circ_X$ above. That said, the following caveats are worth keeping in mind:

1. The dagger structure given by matrix transpose is only going to be meaningful in second-order theories where S is equipped with a trivial action of $\mathbb{Z}_2$ (such as real quantum theory or relational quantum theory). In general second-order theories, the meaningful dagger will be the one defined by the conjugate transpose of matrices—treating the action of $\mathbb{Z}_2$ as conjugation—and the algebras above and below will still be †-Frobenius algebras with respect to this new dagger. In higher-order theories there need not be a meaningful dagger at all.
2. In second order theories, where the traditional CPM construction is employed, the caps from the Frobenius algebras above will be the ones defining the discarding maps. In higher-order theories caps need not be involved in discarding. However, the compact-closed structure still provides maximally-entangled pure states, with the caveat that not all such states need be normalisable.

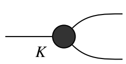
For any group structure $K = (X, \cdot, 1)$ on any finite set X, one also obtains an associated Frobenius algebra $\bullet_K$ on S^X by linearly extending the group multiplication and unit:



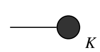
$$= \sum_{x,y\in X} |x\cdot y\rangle \otimes \langle x| \otimes \langle y|$$



$$= |1\rangle$$

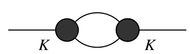


$$= \sum_{x,y\in X} |x\rangle \otimes |y\rangle \otimes \langle x\cdot y|$$



$$= \langle 1| \qquad (3.2)$$

The Frobenius algebra is commutative if and only if the group is and it always satisfies:



$$= \qquad (3.3)$$

Unfortunately, $\bullet_K$ is not quasi-special (a.k.a. normalisable) unless the scalar $|K|$ is multiplicatively invertible: when this is the case, however, we have a legitimate strongly complementary pair $(\circ_X, \bullet_K)$ in S-Mat corresponding to the finite group K. When K is abelian these strongly complementary pairs can be used (under additional constraints) to implement quantum protocols such as the algorithm to solve the abelian Hidden Subgroup Problem [34, 72] or generalised Mermin-type arguments [37, 38]. This also means that certain objects in S-Mat support fragments of the ZX calculus[6] [8, 21], opening the way to the application of well-established diagrammatic techniques.

In ordinary quantum theory, the probabilistic semiring $\mathbb{R}^+$ arises as a sub-semiring of $\mathbb{C}$ fixed by complex conjugation, namely the sub-semiring of those elements $z \in \mathbb{C}$ taking the form $z = x^*x$: this is,

[6] To be precise, they always support the spider rules (but cups/caps for the two algebras are generally distinct), the bialgebra rules, the Hopf laws (with non-trivial antipode), the copy rules, and a generalised version of the π-copy rules (see Ref. [8]). A Hadamard unitary can be defined if and only if the S-valued unitary multiplicative characters for G form a basis for S^X, and in this case the colour-change rules are also supported (taking care to consider Hadamard adjoints where relevant).

essentially, a hallmark of the Born rule. In our more general setting, we can define a G-invariant *norm* on S in the following way:

$$\mathrm{N}(x) := \prod_{\gamma \in G} \varphi(\gamma)(x) \tag{3.4}$$

In ordinary quantum theory, this norm reduces to the Galois field norm $\mathrm{N}(x) := x^* x$ induced by the conjugation symmetry action of $\mathbb{Z}_2$. We could try to define positive elements $z \in S$ to be those in the form $z = \mathrm{N}(x)$, but these are not generally closed under addition. Nevertheless, elements in this form will always be positive, and and element $x \in S$ will be called a *phase* if $\mathrm{N}(x) = 1$.

When classical non-determinism is introduced via the quadratic Born rule associated to the conjugation action of $\mathbb{Z}_2$, quantum theory naturally gives rise to a $\mathbb{R}^+$-probabilistic theory. Similarly, we will prove in Theorem 3.2 below that any quantum-like theory gives rise to an R-probabilistic theory, where R will be the corresponding subset of positive elements (which we will show to be a sub-semiring, in fact).

In its most general case, the higher-order CPM construction of Ref. [39] is rather technical, but the concrete case of $\mathrm{CPM}_{\Phi,\Xi}(S\text{-Mat})$ is straightforward to state. The action Φ of G on S-Mat is obtained by linear extension:

$$\Phi(\gamma)\left(\sum_{y \in Y}\sum_{x \in X} M_x^y |y\rangle\langle x|\right) := \sum_{y \in Y}\sum_{x \in X} \varphi(\gamma)\big(M_x^y\big)\, |y\rangle\langle x| \tag{3.5}$$

From this, one can define the *folding* of a matrix $M := \sum_{y \in Y}\sum_{x \in X} M_x^y |y\rangle\langle x|$ as the following generalisation of *doubling* from the original CPM construction:

$$\mathrm{fld}_\Phi[M] := \bigotimes_{\gamma \in G} \Phi(\gamma)(M) \tag{3.6}$$

Matrices in the form $\mathrm{fld}_\Phi[M]$ are morphisms in $\mathrm{CPM}_{\Phi,\Xi}(S\text{-Mat})$, which we refer to as *pure CP maps*. In particular, *pure states* and *pure effects* are those taking the form $\mathrm{fld}_\Phi[|\psi\rangle]$ and $\mathrm{fld}_\Phi[\langle e|]$ respectively, resulting in the following Born rule (yielding values in the sub-semiring R of positive elements):

$$\mathrm{fld}_\Phi[\langle e|] \circ \mathrm{fld}_\Phi[|\psi\rangle] = \mathrm{N}(\langle e|\psi\rangle) = \prod_{\gamma \in G} \varphi(\gamma)(\langle e|\psi\rangle) \tag{3.7}$$

In the traditional second-order CPM construction, all CP maps are generated as composition of pure CP maps and discarding maps. In higher-order CPM constructions, however, additional effects may be necessary, leading to the consideration of multi-environment structures [39]. A *multi-environment structure* Ξ is, loosely speaking, a collection of G-invariant effects in S-Mat which (i) is closed under tensor product, and (ii) such that the unit scalar 1 is the only effect on the tensor unit included in the collection. The environment structures traditionally studied in CQM are exactly the multi-environment structures with a single effect on each object. Given such a multi-environment structure Ξ, the category $\mathrm{CPM}_{\Phi,\Xi}(S\text{-Mat})$ is defined (modulo some boring technicalities) as the smallest sub-SMC of S-Mat which contains all morphisms in the form $\mathrm{fld}_\Phi[M]$ and all effects in Ξ. We refer to general morphisms in $\mathrm{CPM}_{\Phi,\Xi}(S\text{-Mat})$ as *CP maps*. We refer to the scalars R of $\mathrm{CPM}_{\Phi,\Xi}(S\text{-Mat})$ as the *positive elements* of (S, Φ, Ξ).

Because we are interested in obtaining an R-probabilistic theory—which requires a chosen environment structure to implement discarding—we will require the existence of a distinguished environment structure $\top$ within the multi-environment structure Ξ, i.e. we will require each system to be equipped with a chosen effect in Ξ, the *discarding map*, in a way which is closed under tensor product. A possible choice of discarding maps $\dashv_X$, which we refer to as the *classical discarding maps*, is the one defined as the sum of the effects corresponding to the standard orthonormal basis $(|x\rangle)_{x\in X}$:

$$\dashv_X := \sum_{x\in X} \mathrm{fld}_\Phi\left[\langle x|\right] \tag{3.8}$$

If we use this choice of discarding maps, normalisation of states coincides with the linear extension of the norm on scalars:

$$\dashv_X \circ \mathrm{fld}_\Phi\left[|\psi\rangle\right] = \mathrm{N}\left(|\psi\rangle\right) := \sum_{x\in X} \mathrm{N}\left(\psi^x\right) \qquad \text{where} \qquad |\psi\rangle = \sum_{x\in X} \psi^x |x\rangle \tag{3.9}$$

While the Born rule remains defined by inner products, normalisation in general is not: with the choice of classical discarding maps from Equation 3.8, for example, requiring $\mathrm{N}\left(|\psi\rangle\right) = 1$ is not necessarily the same as requiring $\langle\psi|\psi\rangle = 1$. Whatever the choice of discarding maps, we say that a CP map F is *normalised* if $\dashv \circ F = \dashv$. Under the choice of classical discarding maps from Equation 3.8, combining the discarding maps with the Frobenius algebras $\bigcirc_X$ associated with the standard orthonormal bases $(|x\rangle)_{x\in X}$ yields the following *decoherence maps* on all systems:

$$\mathrm{dec}_{\bigcirc_X} := (id \otimes \dashv_X) \circ \mathrm{fld}_\Phi\left[\quad\right] = \sum_{x\in X} \mathrm{fld}_\Phi\left[|x\rangle\langle x|\right] \tag{3.10}$$

The decoherence maps above are idempotent, so that $(S^X, \mathrm{dec}_{\bigcirc_X})$ appear as objects in the Karoubi envelope $\mathrm{Split}\left(\mathrm{CPM}_{\Phi,\Xi}\left(S\text{-Mat}\right)\right)$ as long as the classical discarding maps of Equation 3.8 appear in the multi-environment structure Ξ. The objects $(S^X, \mathrm{dec}_{\bigcirc_X})$ within the Karoubi envelope were used in Ref. [35] to define classical systems in the ordinary context of mixed-state quantum theory: we will be able to do the same for all quantum-like theories considered here. We can also study how much is lost under decoherence by looking at the group of $\bigcirc$-*phase gates*, the invertible pure CP maps F such that $\mathrm{dec}_\bigcirc \circ F = \mathrm{dec}_\bigcirc$.

Definition 3.1. Let S be a commutative semiring equipped with an action φ of a finite abelian group G. Let Φ be the action of G on S-Mat obtained from φ by linear extension. Let Ξ be a multi-environment structure for Φ on S-Mat such that: (i) every effect $\xi \in \Xi$ has a *normalising state*, i.e. a state $|\bar{\xi}\rangle$ such that $\xi \circ \mathrm{fld}_\Phi\left[|\widehat{\xi}\rangle\right] = 1$; (ii) the effects in Ξ include at least the classical discarding maps from Equation 3.8. The *quantum-like theory* $\mathrm{Quant}_{(S,\Phi,\Xi)}$ is then defined to be the compact-closed symmetric monoidal category $\mathrm{Split}\left(\mathrm{CPM}_{\Phi,\Xi}\left(S\text{-Mat}\right)\right)$. Objects in the form (S^X, id_{S^X}) are called *quantum-like systems*, while objects in the form $(S^X, \mathrm{dec}_{\bigcirc_X})$ are called *classical systems*.

We can show that $\mathrm{Quant}_{(S,\Phi,\Xi)}$ inherits the linear structure of S-Mat, by using ancillas and the classical discarding maps. In particular, the scalars of $\mathrm{Quant}_{(S,\Phi,\Xi)}$, i.e. the positive elements for (S,Φ,Ξ), form a semiring R. This observation finally leads to our main result: quantum-like theories are R-probabilistic, so their operational aspects—such as quantum algorithms and non-locality—are amenable to be studied categorically, compositionally and diagrammatically within the framework of categorical probabilistic theories [35].

Theorem 3.2. *Let* $\mathrm{Quant}_{(S,\Phi,\Xi)}$ *be a quantum-like theory and let R be the sub-semiring of positive elements for (S,Φ). Let* $\dashv$ *be any choice of environment structure on* $\mathrm{Quant}_{(S,\Phi,\Xi)}$*, the* discarding maps *for the theory, which coincides with the classical discarding maps on the classical systems. Under this choice of environment structure,* $\mathrm{Quant}_{(S,\Phi,\Xi)}$ *is an R-probabilistic theory.*

4 Second-order Theories

By a *second-order* quantum-like theory we mean here one where $\varphi : G \to \mathrm{Aut}(S)$ is an action of the group $G = \mathbb{Z}_2$ and the multi-environment structure contains only the discarding maps. We use the lighter notation $\mathrm{Quant}_{(S,^*)}$ for such theories, where * is the involution on S given by φ. Second-order theories are somewhat special: the action of $\mathbb{Z}_2$ is a generalisation of conjugation, which can be used to define an operationally meaningful dagger on S-Mat as the conjugate transpose of matrices. Using this dagger, we can then recover many familiar concepts from quantum theory: isometries, unitaries, inner products and the traditional quadratic form of the Born rule. The CPM construction reduces to the traditional one from Ref. [65] and the environment structure is given by the cap. In fact, a recent reconstruction result [71] shows that every dagger compact theory satisfying certain principles[7] is necessarily a second-order theory for some ring S. Second-order theories are also compatible with the framework developed in Ref. [11], which can therefore be used to investigate their natural notions of computational complexity.

In the second-order theories which follow, we are primarily interested in understanding the structure of the group of phases and its impact on computation and non-locality. As our use case for its impact on computation we will consider the algorithm for the abelian Hidden Subgroup Problem (HSP) [34]. As our use case for its impact on non-locality we will look at Mermin-type arguments [38].

4.1 Ordinary quantum theory

Ordinary *quantum theory* is the second-order theory $\mathrm{Quant}_{(\mathbb{C},^*)}$ given by considering the field $\mathbb{C}$ of complex numbers equipped with complex conjugation *. The *probability semiring* $\mathbb{R}^+$ is the sub-semiring of positive elements for $(\mathbb{C},^*)$, so that $\mathrm{Quant}_{(\mathbb{C},^*)}$ is a probabilistic theory. The group of phases in $(\mathbb{C},^*)$—relevant to the Fourier transform and the algorithm to solve the HSP—is the circle group T^1. Consider a classical structure $\circ$ on $\mathbb{C}^d$: the group of $\circ$-phase gates—relevant to Mermin-type non-locality arguments—is isomorphic to the $(|X|-1)$-dimensional torus $T^{|X|-1}$. The abelian HSP can be solved efficiently in quantum theory for all finite abelian groups. All Mermin-type arguments can be implemented in quantum theory [37], proving that the theory is non-local.

4.2 Real quantum theory

The simplest non-conventional example is given by the ring $\mathbb{R}$ of signed reals (with the trivial involution), which yields the *probability semiring* $\mathbb{R}^+$ as its sub-semiring of positive elements; in particular, all positive elements are pure scalars. The corresponding probabilistic theory $\mathrm{Quant}_{(\mathbb{C},id)}$ is known as *real quantum theory* [10, 12, 47, 74]: it is arguably the most well-studied of the quantum-like theories, and the closest to ordinary quantum theory. The group of phases in real quantum theory is $\{\pm 1\} \cong \mathbb{Z}_2$, making non-trivial interference possible: in particular, Simon's problem and other Hidden Subgroup Problems on $\mathbb{Z}_2^N$ can all be solved efficiently in real quantum theory. Consider a classical structure $\circ$ on $\mathbb{R}^d$ with enough classical states, which corresponds to an orthonormal basis of $\mathbb{R}^d$ (because $\mathbb{R}$ is multiplicatively cancellative [29]). The group of $\circ$-phase gates is isomorphic to the group $\mathbb{Z}_2^{d-1}$ of $(d-1)$-bit strings under bitwise xor. Because of the structure of phase groups, generalised Mermin-type arguments only yield local empirical models [37]. Nevertheless, Bell's theorem goes through in real quantum theory (as it only involves states and measurements on the ZX great circle of the Bloch sphere), which is therefore a non-local probabilistic theory.

[7] Namely *strong purification*, *existence of dagger kernels*, *pure exclusion*, *conditionaning* and *boundedness* of scalars.

4.3 Relational quantum theory

Examples of an entirely different nature are given by considering distributive lattices Ω (with the trivial involution), which yield themselves back as their sub-semirings of positive elements (because of multiplicative idempotence). Distributive lattices seem to be almost as far as one can get from the probabilistic semiring $\mathbb{R}^+$, but the category Ω-Mat has been studied extensively as a toy model for quantum theory (especially in the boolean case $\Omega = \mathbb{B}$) [6, 23, 32, 59, 76]. The corresponding second-order theory $\mathrm{Quant}_{(\Omega,id)}$ is known as *relational quantum theory*, it is *possibilistic* and the special boolean case $\Omega = \mathbb{B}$ has been studied in [33, 53]. The group of phases in Ω is the singleton $\{1\}$, and no interference is possible in relational quantum theory. Relational quantum theory also feature very few quantum-to-classical transitions: there is a unique basis on each system, namely the one given by the elements of the underlying set. The theory is locally tomographic on pure states, but they fails to be tomographic altogether on mixed states: for example, the pure state $|\psi\rangle\langle\psi|$ for $|\psi\rangle := |0\rangle + |1\rangle$ and the mixed state $|0\rangle\langle 0| + |1\rangle\langle 1|$ are distinct, but cannot be distinguished by measurement. In fact, a characteristic trait of relational quantum theory is exactly that superposition and mixing are essentially indistinguishable (because of idempotence) [6,33,53]. Classical structures $\circ$ in relational quantum theory over the booleans are known to correspond to abelian groupoids $\oplus_{i\in I} G_i$ [59], and the corresponding group of $\circ$-phase gates is isomorphic to $\prod_{i\in I} G_i$. It can be shown that generalised Mermin-type arguments only yield local empirical models [37]. In fact, it can be shown that that relational quantum theory is entirely local [6, 33].

4.4 Hyperbolic quantum theory

Turning our attention back to real algebras, we can consider the commutative ring of *split complex numbers* $\mathbb{R}[\sqrt{1}] := \mathbb{R}[X]/(X^2-1)$, a two-dimensional real algebra. Split complex numbers take the form $(x + jy)$, where $x, y \in \mathbb{R}$ and $j^2 = 1$; in particular, they have non-trivial zero-divisors in the form $a(1 \pm j)$, because $(1 + j)(1 - j) = 1 - j^2 = 0$. They come with the involution $(x + jy)^* := x - jy$, which yields the *signed-probability ring* $\mathbb{R}$ as sub-semiring of positive elements. We refer to the corresponding $\mathbb{R}$-probabilistic theory $\mathrm{Quant}_{(\mathbb{R}[\sqrt{1}],^*)}$ as *hyperbolic quantum theory*[8] [50, 51, 55].

Hyperbolic quantum theory is an extremely interesting theory. On the one hand, it contains real quantum theory as a sub-theory[9], and as a consequence every scenario and protocol which can be implemented in real quantum theory (such as the algorithm to efficiently solve Simon's problem [34]) can also be implemented in hyperbolic quantum theory. On the other hand, hyperbolic quantum theory is a local theory, in the sense that every empirical model arising in hyperbolic quantum theory admits a local hidden variable model in terms of signed probabilities (the notion of classical non-determinism for hyperbolic quantum theory) [2]. While signed probabilities might at first sound unphysical, an operational interpretation exists in terms of unsigned probabilities on signed events [3][10].

The group of phases in $\mathbb{R}[\sqrt{1}]$ consists of the elements with square norm 1, i.e. the elements in the form $x + jy$ which lie on the following unit hyperbola of the real plane:

$$1 = (x + jy)^*(x + jy) = x^2 - y^2 \tag{4.1}$$

In fact, the natural geometry for the split complex numbers is that of the real plane endowed with the Lorentzian metric $-dy^2 + dx^2$, i.e. that of the Minkowski plane. Just like multiplication by phases in

[8] Clifford referred to functions of split complex numbers as "functions of a motor variable" [19], so we could say that hyperbolic quantum theory is the theory of *wavefunctions of a motor variable* (how does *motor quantum theory* sound?).

[9] By which we mean that $\mathbb{R}[\sqrt{1}]$ contain $\mathbb{R}$ as a sub-ring fixed by the involution.

[10] Where the sign of the events themselves cannot be observed, yielding an epistemic restriction which could be seen as not-too-far-removed from the one which originally motivated Spekkens's toy model [15, 69]

$\mathbb{C}$ forms the circle group $U(1)$ of rotations around the origin for the Euclidean plane, multiplication by phases in $\mathbb{R}[\sqrt{1}]$ forms the group $SO(1,1)$ of orthochronous homogeneous Lorentz transformations for the Minkowski plane. We have the isomorphism of Lie groups $\mathbb{Z}_2 \times \mathbb{R} \cong SO(1,1)$ given by $(s,\theta) \mapsto s(\cosh(\theta) + j\sinh(\theta))$: as a consequence, the $\mathbb{R}[\sqrt{1}]$-valued multiplicative characters for finite groups are exactly the same as the $\mathbb{R}$-valued multiplicative characters, and the only finite groups with enough multiplicative characters to form a Fourier basis are the ones in the form $\mathbb{Z}_2^N$; Simon's problem, and other Hidden Subgroup Problems for $\mathbb{Z}_2^N$, can be efficiently solved in hyperbolic quantum theory, despite the latter being local. Things are different for infinite groups such as $\mathbb{Z}$, which have enough $\mathbb{R}[\sqrt{1}]$-valued multiplicative characters but not enough $\mathbb{R}$-valued multiplicative characters.

Now consider a classical structure $\circ$ corresponding to an orthonormal basis of $\mathbb{R}[\sqrt{1}]^d$ (we have to ask explicitly for orthogonality, because the result of [29] does not apply to hyperbolic quantum theory: $\mathbb{R}[\sqrt{1}]$ has non-trivial zero-divisors, and hence it is not multiplicatively cancellative). The group of $\circ$-phase gates is isomorphic to $(\mathbb{Z}_2 \times \mathbb{R})^{d-1}$, and has $\mathbb{Z}_2^{d-1}$ as a maximal finite subgroup: as a consequence, generalised Mermin-type arguments (which involve finite groups) only yield local empirical models, just as in real quantum theory. However, extensions of Mermin-type arguments to infinite groups yield different results: this is because subgroups like $(\{0\} \times \mathbb{Z}) \trianglelefteq (\mathbb{Z}_2 \times \mathbb{R})$ would become available, and there are equations (such as $2\theta = 1$) which have no solutions in the subgroup $\{0\} \times \mathbb{Z}$ but have solutions (e.g. $\theta = (0, \frac{1}{2})$ and $\theta = (1, \frac{1}{2})$) in the larger group $\mathbb{Z}_2 \times \mathbb{R}$.

4.5 p-adic quantum theory

We now look at the construction of *p-adic quantum mechanics* [48,49,56,57,61,73], where $R := Q_p$ is the field of p-adic numbers, and S is some quadratic extension. Here, we use the notation Q_p to denote the p-adic numbers, and Z_p to denote the p-adic integers, to distinguish them from the finite field $\mathbb{Z}_p$ of integers modulo p; note that this convention is different from the one used in many texts on p-adic arithmetic, where $\mathbb{Z}_p$ is used for the p-adic integers (and $\mathbb{Q}_p$ for the p-adic numbers).

When $p > 2$, the p-adic numbers $x := p^{\operatorname{ord} x} \sum_{i=0}^{+\infty} x_i p^i$ fall within four distinct quadratic classes—jointly labelled by the parity of the order $\operatorname{ord} x \in \mathbb{Z}$ and by the quadratic class of the first non-zero digit $x_0 \in \mathbb{Z}_p^\times$—corresponding to three inequivalent quadratic extensions. This means that there is no way to obtain all positive elements as pure scalars by a single quadratic extension. This would seem to indicate that mixed states play a necessary role in the emergence of p-adic probabilities, which cannot all be obtained from pure states alone: the potential physical significance of this observation might become the topic of future work on p-adic quantum theory within CQM.

We consider the quadratic extension $S := Q_p(\sqrt{\varepsilon})$, where $p \geq 3$ and ε is a primitive element in the field $\mathbb{Z}_p$ of integers modulo p, and we follow the presentation of Ref. [61]. A generic element of $Q_p(\sqrt{\varepsilon})$ takes the form $c + s\sqrt{\varepsilon}$, for $c, s \in Q_p$, and its norm is $\mathrm{N}\left(c + s\sqrt{\varepsilon}\right) := |c + s\sqrt{\varepsilon}|^2 = (c - s\sqrt{\varepsilon})(c + s\sqrt{\varepsilon}) = c^2 - \varepsilon s^2$. We define conjugation as $(c + s\sqrt{\varepsilon})^* := (c - s\sqrt{\varepsilon})$ and we refer to $\operatorname{Quant}_{(Q_p(\sqrt{\varepsilon}),^*)}$ as *p-adic quantum theory*. Whether an element $x \in Q_p$ can be written in norm form, i.e. whether is is a pure scalar in p-adic quantum theory, is determined by the *sign function* $\operatorname{sgn}_\varepsilon x$, which takes the value $+1$ if $x = c^2 - \varepsilon s^2$ for some $c, s \in Q_p$, and the value -1 otherwise. An explicit form for the sign function (in the $p \neq 2$ case) is given by Equation (2.34) of Ref. [61], which specialised to our case ($\tau = \varepsilon$ and $\operatorname{ord} \tau = 0$) reads $\operatorname{sgn}_\varepsilon x = (-1)^{\operatorname{ord} x}$. Hence the pure scalars in $\mathrm{CP}^*[Q_p(\sqrt{\varepsilon})\text{-Mat}]$ are exactly the p-adic numbers x with even order $\operatorname{ord} x$; closure of this set under addition yields $R := Q_p$ as sub-semiring (field, in fact) of positive elements in $S := Q_p(\sqrt{\varepsilon})$. Hence p-adic quantum theory has the p-adic numbers as its natural notion of classical non-determinism, as expected.

The phases in p-adic quantum theory are those $\xi := (c + s\sqrt{\varepsilon}) \in Q_p(\sqrt{\varepsilon})$ such that $\xi^* \xi = c^2 - \varepsilon s^2 = 1$.

In Ref. [61] (Equation (4.35) of Section IV.C, and Equation (C12b) of Appendix C.3) it is shown that phases form a multiplicative group C_ε isomorphic to the additive group $\mathbb{Z}_{p+1} \times pZ_p$—here $(\mathbb{Z}_{p+1}, +, 0)$ are the integers modulo $p+1$, while $(pZ_p, +, 0)$ is the additive subgroup of Z_p formed by those p-adic integers which are divisible by p. In particular, C_ε is an infinite group with the cardinality of the continuum, and each "sheet" pZ_p is a profinite[11] torsion-free group, which is best understood by looking at the descending normal series $pZ_p \rhd p^2Z_p \rhd \ldots \rhd p^mZ_p \rhd \ldots$ and considering the finite cyclic quotients $p^nZ_p/p^mZ_p \cong \mathbb{Z}_{p^{m-n}}$.

The scalar $|G|$ is always invertible, and it is in the form $|G| = z_G^* z_G$ if and only if the largest power of p which divides $|G|$ is even. Furthermore, G has enough $Q_p(\sqrt{\varepsilon})$-valued multiplicative characters if and only if $G \cong \prod_{k=1}^{K} \mathbb{Z}_{p_k^{e_k}}$ with $p_k^{e_k} | p+1$ for all $k = 1, ..., K$ (in the light of Hensel's Lemma, this parallelism between p-adic quantum theory and finite-field quantum theory on $\mathbb{F}_p$, presented in the Appendix, should not come as a big surprise): finite abelian groups G satisfying this condition admit efficient solutions for Hidden Subgroup Problems in p-adic quantum theory (because we necessarily have that p cannot divide $|G|$). Similarly, it is possible to formulate non-trivial generalised Memrin-type arguments in p-adic quantum theory if and only if $p+1$ is not square-free. That said, p-adic quantum theory is a local theory by virtue of Theorem 2.3.

Remark 4.1. Similar considerations apply to the the construction of p-adic quantum theory for the other two quadratic extensions $Q_p(\sqrt{p})$ and $Q_p(\sqrt{p\varepsilon})$ available in the case of $p \geq 3$ (although the cases $p = 3$ and $p \geq 5$ have to be treated separately), as well as the seven quadratic extensions available in the case of $p = 2$. The phase groups take a similar (but not identical) form to the one presented here, and the full details can be found in Ref. [61] (Section IV.C and Appendices C.3, C.4).

5 Higher-order Theories

Second-order theories share many of the familiar features of quantum theory because of way the Born rule is formulated in terms of inner products. As such, they also share many of the limitations of quantum theory, notably the impossibility to display effects such as higher-order interference or hyper-decoherence, which have recently become the subject of no-go results [13, 52]. By moving from the usual CPM construction to higher-order CPM constructions—i.e. by considering finite abelian groups G different from $\mathbb{Z}_2$ and allowing more than one environment structure to be used in the COM construction—a new universe of theories becomes available.

Below we describe two such constructions, known as *double-dilation* and *double-mixing* [30]. Though originally conceived within the framework of compositional distributional semantics, double-dilation of $\mathbb{C}$-Mat has been shown [40] to be a probabilistic theory displaying higher-order interference effects and possessing hyper-decoherence maps down to quantum theory.

5.1 Double-dilation

We can equip the complex numbers $\mathbb{C}$ with the following action $\varphi : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{C})$ of the finite abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2$:

$$\begin{array}{llll} \varphi(0,0) := z \mapsto z & \qquad \varphi(0,1) := z \mapsto z^* \\ \varphi(1,0) := z \mapsto z^* & \qquad \varphi(1,1) := z \mapsto z \end{array} \tag{5.1}$$

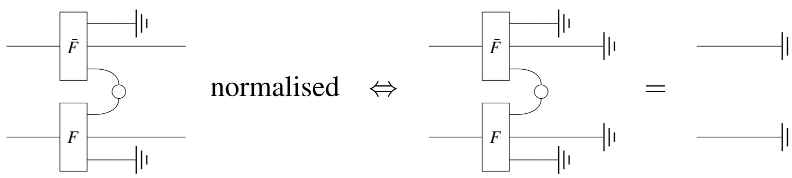
This action extends the usual conjugation action of $\mathbb{Z}_2$, so we can always represent $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-folded morphisms from $\mathbb{C}$-Mat as $\mathbb{Z}_2$-folded morphisms from the usual CPM construction CPM$(\mathbb{C}$-Mat$)$, i.e. as

[11] And hence both compact and totally disconnected.

doubled CP maps from ordinary quantum theory. Our choice of multi-environment structure will also be such that all effects are CP maps from ordinary quantum theory, so that we can use the diagrammatic calculus of CPM $(\mathbb{C}\text{-Mat})$ to reason about this theory:

(5.2)

The multi-environment structure Ξ is generated by the two basic effects presented above. The construction is known as *double-dilation* [30], and we refer to the fourth-order theory $\mathrm{Quant}_{(\mathbb{C},\Phi,\Xi)}$ as the *theory of density hypercubes* [40]. The discarding maps for this theory are chosen to be those of CPM $(\mathbb{C}\text{-Mat})$:

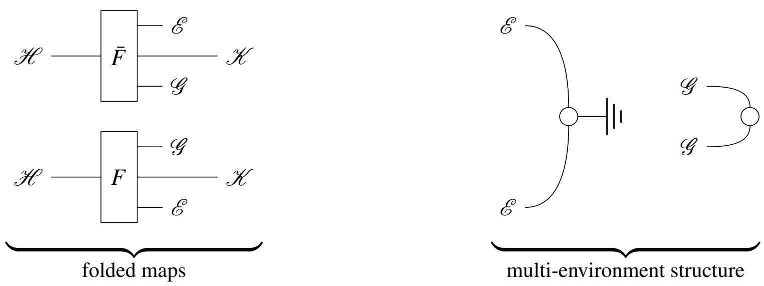


(5.3)

In particular, these are *not* the classical discarding maps from Equation 3.8, though the latter appear in Ξ. It is possible to show that the theory of density hypercubes displays higher-order interference effects and that quantum theory arises from density hypercubes by a mechanism analogous to decoherence [40], known as hyper-decoherence. Higher-order interference is linked to computational advantage in certain tasks [14]. The display of higher-order interference and hyper-decoherence is also interesting from a foundational perspective, since recent no-go results [13, 52] ruled out such possibility in a large class of probabilistic theories.

5.2 Double-mixing

A variation of double-dilation known as *double-mixing* is also studied in [30]. Folding is the same, but the discarding maps of double-dilation are replaced by the classical discarding maps from Equation 3.8:



(5.4)

The quantum-like theory thus obtained can be understood within the theory of density hypercubes: it consists exactly of those CP maps of density hypercubes which are normalised with respect to hyper-decoherence to quantum theory (which, contrary to ordinary decoherence to classical theory, cannot be performed deterministically). It is not presently known whether higher-order interference effects also appear in the double-mixing construction, though the no-go result of [52] seems to suggest they will not.

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A New second-order quantum-like theories

A.1 Parity quantum theory

A simple variation on relational quantum theory (over the booleans) is given by using symmetric difference of sets, instead of union, as the superposition operation. This leads us to consider the finite field with two elements $\mathbb{Z}_2 := (\{0,1\},+,0,\times,1)$, with trivial involution, in place of the booleans $\mathbb{B} := (\{0,1\},\vee,0,\times,1)$, also with trivial involution. The multiplication is the same, but addition is now non-idempotent, and superposition is no longer the same as mixing. The *parity semiring* $\mathbb{Z}_2$ yields itself back as its sub-semiring of positive elements and we refer to the corresponding $\mathbb{Z}_2$-probabilistic theory $\mathrm{Quant}_{(\mathbb{Z}_2,id)}$ as *parity quantum theory*.

Remark A.1. Parity quantum theory as defined here (the same as in Ref. [11]) pretty much coincides with the $\mathbb{Z}_2$ case of modal quantum theory [63, 64], but it should be noted that the philosophical interpretation of $\mathbb{Z}_2$-valued probabilities is significantly different. In modal quantum theory, the interest is in generating possibilistic tables by using finite fields, subsequently interpreting all zero values as the boolean 0 and all non-zero values as the boolean 1. In parity quantum theory, the non-determinism itself is interpreted to be natively $\mathbb{Z}_2$-valued, and no attempt is made to translate the resulting empirical models into possibilistic ones. Indeed, such an interpretation would not be natural within our semiring-oriented framework, as no semiring homomorphism can exists from any finite field into the booleans.

The group of phases in $\mathbb{Z}_2$ is the singleton $\{1\}$, but interference is still possible in parity quantum theory: this somewhat counter-intuitive situation is made possible by the fact that 1 is its own additive inverse in $\mathbb{Z}_2$, so that triviality of the group of phases is slightly deceptive. Indeed, consider the four two-qubit states below, which form an orthonormal basis for $\mathbb{Z}_2^2$:

$$\begin{aligned} |\psi_{012}\rangle &:= |00\rangle + |01\rangle + |10\rangle & |\psi_{123}\rangle &:= |01\rangle + |10\rangle + |11\rangle \\ |\psi_{230}\rangle &:= |10\rangle + |11\rangle + |00\rangle & |\psi_{301}\rangle &:= |11\rangle + |00\rangle + |01\rangle \end{aligned} \tag{A.1}$$

For example, we have $|10\rangle = |\psi_{012}\rangle + |\psi_{123}\rangle + |\psi_{230}\rangle$. When measured in the computational basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, the normalised states $|01\rangle$, $|10\rangle$ and $|\psi_{012}\rangle$ all have non-zero $\mathbb{Z}_2$-probability of yielding an outcome in the set $\{01, 10\}$, but their superposition $|01\rangle + |10\rangle + |\psi_{012}\rangle = |00\rangle$ (also a normalised state) has zero $\mathbb{Z}_2$-probability of yielding an outcome in that set.

Because the group of phases is trivial, so are all the groups of phase gates, as well as all the $\mathbb{Z}_2$-valued multiplicative characters of all groups; as a consequence, parity quantum theory admits no non-trivial generalised Mermin-type arguments, and no implementation of the algorithm to solve the HSP. Furthermore, Theorem 2.3 shows that parity quantum theory is local, because $\mathbb{Z}_2$ is a field.

R-probabilistic theories can be similarly constructed for modal quantum theory over any other finite field $\mathbb{F}_{p^n}$ [63, 64], by taking $S := \mathbb{F}_{p^n}$ with the trivial involution. However, these theories have a lot of non-pure scalars—namely the $(p^n-1)/2$ non-squares in $\mathbb{F}_{p^n}$—and their phases are close to trivial—namely they are $\{\pm 1\}$ if $p > 2$ and $\{1\}$ if $p = 2$. Instead, we will consider a more sophisticated construction based on quadratic extensions of finite fields, which we will refer to as "finite-field quantum theory".[12] Finite-field quantum theory is a local theory (by Theorem 2.3), in which it is nonetheless possible to formulate non-trivial quantum algorithms, as well as non-trivial Mermin-type "non-locality" arguments. This is in stark contrast with more traditional toy models such as Spekkens's toy model [15, 31, 69] and relational quantum theory, in which the quantum Fourier transform cannot be performed for non-trivial groups [36] (precluding the implementation of algorithms based on it), and in which all Mermin-type arguments are necessarily trivial [22, 24, 37].

A.2 Finite-field quantum theory

Consider a finite field $\mathbb{F}_{p^n}$ (with p odd), and let ε be a generator for the cyclic group $\mathbb{F}_{p^n}^\times$ of invertible (aka non-zero) elements in $\mathbb{F}_{p^n}$ (i.e. a primitive element for $\mathbb{F}_{p^n}$). We consider the ring $\mathbb{F}_{p^n}[\sqrt{\varepsilon}] := \mathbb{F}_{p^n}[X^2-\varepsilon]$, equipped with the involution $(x+y\sqrt{\varepsilon})^* := (x-y\sqrt{\varepsilon})$: because ε is a primitive element, $\mathbb{F}_{p^n}(\sqrt{\varepsilon}) \cong \mathbb{F}_{p^{2n}}$ is a field. We are in fact working with the quadratic extension of fields $\mathbb{F}_{p^n}(\sqrt{\varepsilon})/\mathbb{F}_{p^n}$, equipped with the usual involution and norm from Galois theory:

$$\mathrm{N}\left(x+y\sqrt{\varepsilon}\right) := \left|x+y\sqrt{\varepsilon}\right|^2 = (x-y\sqrt{\varepsilon})(x+y\sqrt{\varepsilon}) = x^2-\varepsilon y^2 \tag{A.2}$$

The sub-field $\mathbb{F}_{p^n}$ (given by the elements in the form $x+0\sqrt{\varepsilon}$) is the sub-semiring of positive elements (and we will shortly see that all positive elements are pure scalars). We refer to the $\mathbb{F}_{p^n}$-probabilistic theory $\mathrm{Quant}_{(\mathbb{F}_{p^n}[\sqrt{\varepsilon}],*)}$ as *finite-field quantum theory*.

The phases in $\mathbb{F}_{p^n}(\sqrt{\varepsilon})$ are the points (x,y) of the $\mathbb{F}_{p^n}^2$ plane lying on the unit hyperbola $x^2-\varepsilon y^2 = 1$, which does not factor as a product of two lines because ε is a primitive element. The following iconic result of Galois theory, due to Hilbert, can be used to characterise them (see e.g. Ref. [46] for a proof).

Theorem A.2 (*Hilbert's Theorem 90*).
Let L/K be a finite cyclic field extension, and let $\sigma : L \to L$ be a generator for its cyclic Galois group. Then the multiplicative group of elements $\xi \in L$ of unit relative norm $N_{L/K}(\xi) = 1$ is isomorphic to the quotient group $L^\times/K^\times$.

[12] A related construction features in Ref. [11], but from a computational complexity angle rather than a physical theory one.

Corollary A.3. *The phases in* $\mathbb{F}_{p^n}(\sqrt{\varepsilon})$ *form the cyclic group* $\mathbb{F}_{p^{2n}}^{\times}/\mathbb{F}_{p^n}^{\times} \cong \mathbb{Z}_{p^n+1}$.

Another interesting consequence of Hilbert's Theorem 90 is the fact that the positive elements in finite-field quantum theory are all pure scalars.

Lemma A.4. *All scalars in finite-field quantum theory are pure.*

We have seen that finite-field quantum theory comes with a non-trivial phase group, which in turn allows for non-trivial implementations of certain quantum protocols. We open with a result about the Quantum Fourier Transform, which combined with the main result of Ref. [34] implies that the Hidden Subgroup Problem can be solved efficiently in finite-field quantum theory for arbitrarily large families of finite abelian groups (as p^n grows larger).

Lemma A.5. *Let G be a finite abelian group. Then G has enough* $\mathbb{F}_{p^n}(\sqrt{\varepsilon})$*-valued unitary multiplicative characters if and only if* $G \cong \prod_{k=1}^{K} \mathbb{Z}_{p_k^{e_k}}$ *with* $p_k^{e_k} | p^n+1$ *for all* $k = 1,\ldots,K$. *When this is the case, the Hidden Subgroup Problem for G can be solved efficiently in finite-field quantum theory.*

Now consider a classical structure $\bigcirc$ with enough classical states on a d-dimensional quantum system in finite-field quantum theory, which corresponds to an orthonormal basis of the vector space $\left(\mathbb{F}_{p^n}(\sqrt{\varepsilon})\right)^d$ (because $\mathbb{F}_{p^n}(\sqrt{\varepsilon})$ is multiplicatively cancellative [29]). Then the group of $\bigcirc$-phase gates in $\mathrm{CP}^*[\mathbb{F}_{p^n}(\sqrt{\varepsilon})\text{-Mat}]$ is isomorphic to the group $\mathbb{Z}_{p^n+1}^{d-1}$.

Lemma A.6. *It is possible to formulate non-trivial generalised Mermin-type arguments*[13] *in finite-field quantum theory if and only if* p^n+1 *is not a square-free natural number.*

As mentioned before, finite-field quantum theory is a local theory, by virtue of Theorem 2.3. While finite-field quantum theory and parity quantum theory might not have as direct a physical interpretation as hyperbolic quantum theory and relational quantum theory, they come with the major advantage of having wavefunction valued over a field, so that objects are finite-dimensional vector spaces (equipped with a non-standard inner product, in the case of finite-field quantum theory). This opens the door for a systematic study of quantum systems in these theories using standard tools from finite geometry. Further investigation in this direction is left to future work.

A.3 Tropical quantum theory

Relational quantum theory involves semirings which are both additively and multiplicatively idempotent, parity quantum theory involves a semiring which is only multiplicatively idempotent, and ordinary quantum theory involves a semiring which is neither additively nor multiplicatively idempotent. We now give examples of theories with wavefunctions based in semirings which are additively idempotent but not multiplicatively idempotent, namely the tropical semirings [54, 58, 67, 68, 70].

Definition A.7. A *tropical semiring* is the commutative semiring $(M, \min, \infty, +, 0)$ obtained from a totally ordered commutative monoid $(M, +, 0, \leq)$ having an absorbing element ∞ which is larger than all elements in the monoid. In the tropical semiring, min is the addition, ∞ is the additive unit, $+$ is the multiplication and 0 is the multiplicative unit. The nomenclature is extended to semirings isomorphic to the explicitly min-plus semirings used above (e.g. max-plus formulations, or the Viterbi semiring).

Examples of tropical semirings in the literature include the tropical reals $(\mathbb{R} \sqcup \{\infty\}, \min, \infty, +, 0)$, the tropical integers $(\mathbb{Z} \sqcup \{\infty\}, \min, \infty, +, 0)$, the tropical naturals $(\mathbb{N} \sqcup \{\infty\}, \min, \infty, +, 0)$, and the Viterbi semiring $([0,1], \max, 0, \cdot, 1)$ (which is a tropical semiring because it is isomorphic to the explicitly min-plus semiring $(\mathbb{R}^+ \sqcup \{\infty\}, \min, \infty, +, 0)$ via the semiring homomorphism $x \mapsto -\log x$). In fact, there is an

[13] By *non-trivial* we mean arguments for systems of equation having no solutions in the subgroup of classical states.

easy characterisation of which commutative semirings arise as tropical semirings (the proof is omitted as it is a straightforward check).

Lemma A.8. *A commutative semiring $(S,+,0,\cdot,1)$ is a tropical semiring if and only if for all $a,b \in S$ we have $a = a+b$ or $b = a+b$ (in which case we can set $\min(a,b) = a+b$).*

From now on, we will revert back to usual semiring notation, and we will rely on the result above to connect with the min-plus notation typical of tropical geometry [70]. We will, however, remember that tropical semirings come with a total order respected by the multiplication, and we will occasionally use min, max and $\leq$ in addition to the addition/multiplication.

Lemma A.9. *The only involution possible on a tropical semiring $(S,+,0,\cdot,1)$ is the trivial one, and the positive elements form the sub-semiring of squares $(\{x^2 \,|\, x \in S\},+,0,\cdot,1)$.*

If S is a tropical semiring and $R := (\{x^2 \,|\, x \in S\},+,0,\cdot 1)$ is its sub-semiring of positive elements, we refer to the R-probabilistic theory $\mathrm{Quant}_{(S,id)}$ as *tropical quantum theory*. Just as in the case of relational quantum theory, the group of phases in a tropical semiring S is always trivial (because $x^2 = 1$ implies $x = 1$ in any totally ordered monoid $(S,\cdot,1,\leq)$), so there is no interference. Similarly, there is a unique orthonormal basis on each system, the only unitaries/invertible maps are permutations, and superposition cannot be distinguished from mixing by measurements alone. Tropical quantum theory does not admit any implementation of the algorithm for the abelian Hidden Subgroup Problem, nor does it admit any generalised Mermin-type non-locality arguments.

The parallelisms with relational quantum theory become less surprising when one realises that tropical quantum theory is another generalisation of quantum theory over the booleans: the latter form a totally ordered distributive lattice, and hence are a particular case of tropical semiring. (Proof of the following result is omitted, as it is a straightforward check.)

Lemma A.10. *Any totally ordered distributive lattice $(\Omega,\vee,\bot,\wedge,\top)$ is a tropical semiring $(\Omega,\wedge,\top,\vee,\bot)$; conversely, every tropical semiring $(S,+,0,\cdot,1)$ which has 1 as least element and such that $x^2 = x$ for all $x \in S$ is a totally ordered distributive lattice $(S,\cdot,1,+,0)$.*

In the light of the result above, we expect tropical quantum theory to be local, exactly like relational quantum theory, but further investigation of this question is left to future work.

B Proofs

Proof of Theorem 2.3. Theorem 5.4 from Ref. [2] states that all no-signalling empirical models over the field $\mathbb{R}$ admit a local hidden variable model in terms of signed probabilities. Although the original result was proven for $\mathbb{R}$, close inspection reveals that it holds for no-signalling empirical models over any field k: as a consequence, Bell-type measurement scenarios in R-probabilistic theories where R is a field give rise to no-signalling empirical models admitting local hidden variable models. Finally, R-probabilistic theories have a sub-SMC of finite R-probabilistic classical systems, with all R-distributions as normalised states and all R-stochastic maps as normalised processes: as a consequence, all local hidden variable models valued in R can be realised in any and all R-probabilistic theories. □

Proof of Theorem 3.2. In order for $\mathrm{Quant}_{(S,\Phi,\Xi)}$ to be R-probabilistic under the CMon-enrichment of S-Mat, we need to show that it satisfies the following three conditions:

(i) there is a full sub-SMC $(\mathrm{Quant}_{(S,\Phi,\Xi)})_K$ of $\mathrm{Quant}_{(S,\Phi,\Xi)}$ that is equivalent to R-Mat;

(ii) the CMon-enrichment of S-Mat must restrict to a well-defined CMon-enrichment for $\mathrm{Quant}_{(S,\Phi,\Xi)}$, which coincides on $(\mathrm{Quant}_{(S,\Phi,\Xi)})_K$ with the enrichment of R-Mat;

(iii) the SMC $\mathrm{Quant}_{(S,\Phi,\Xi)}$ comes with an environment structure which restricts to the the canonical one from R-Mat on the full subcategory $(\mathrm{Quant}_{(S,\Phi,\Xi)})_K$.

The key is to show that the CMon-enrichment of S-Mat restricts to a well-defined CMon-enrichment for $\mathrm{Quant}_{(S,\Phi,\Xi)}$. To do so, we can use the classical discarding maps, which we are guaranteed to have in Ξ. Firstly, note that if $(M^{(z)})_{z\in Z}$ is a family of matrices then we can always obtain $\sum_{z\in Z} \mathrm{fld}_\Phi\left[M^{(z)}\right]$ using classical discarding maps:

$$\sum_{z\in Z} \mathrm{fld}_\Phi\left[M^{(z)}\right] = (id \otimes \dashv\!\shortparallel_Z)\circ \mathrm{fld}_\Phi\left[\sum_{z\in Z} M^{(z)}\otimes |z\rangle\right] \tag{B.1}$$

Hence, all we need to do is show that we can also obtain linear combinations of effects in Ξ. If $(\xi^{(z)})_{z\in Z}$ is a family of effects in Ξ, without loss of generality on the same object S^X of $\mathrm{CPM}_{\Phi,\Xi}(S\text{-Mat})$, then we can consider the following matrix $S^X \to (\otimes_{z\in Z} S^X)\otimes S^Z$:

$$F := \sum_{z\in Z}\sum_{x\in X}\left(\bigotimes_{z'\in Z}|e(x,z,z')\rangle\right)\otimes |z\rangle \tag{B.2}$$

where $|e(x,z,z')\rangle$ is defined to be $|x\rangle$ if $z=z'$ and $|\widehat{\xi^{(z')}}\rangle$ otherwise. We can use this matrix to obtain the sum of the effects:

$$\sum_{z\in Z}\xi^{(z)} = \left(\bigotimes_{z\in Z}\xi^{(z)}\right)\otimes \dashv\!\shortparallel_z \circ \mathrm{fld}_\Phi[F] \tag{B.3}$$

For condition (i), consider the full-subcategory $(\mathrm{Quant}_{(S,\Phi,\Xi)})_K$ of $\mathrm{Quant}_{(S,\Phi,\Xi)}$ spanned by those objects in the form $(S^X, \mathrm{dec}_{\bigcirc_X})$, where X is a finite set, $\bigcirc_X$ is the special commutative Frobenius algebra on S^X associated with the orthonormal basis $|x\rangle_{x\in X}$, and $\mathrm{dec}_{\bigcirc_X} : S^X \to S^X$ is the decoherence map for $\bigcirc_X$ obtained using the classical discarding maps. Morphisms $(S^X, \mathrm{dec}_{\bigcirc_X}) \to (S^Y, \mathrm{dec}_{\bigcirc_Y})$ are exactly those in the following form, where $(f_{xy})_{x\in X, y\in Y}$ is an arbitrary matrix of scalars (i.e. elements of R):

$$\sum_{y\in Y}\sum_{x\in X}\mathrm{fld}_\Phi[|y\rangle]\ f_{xy}\ \mathrm{fld}_\Phi[\langle x|] \tag{B.4}$$

As a consequence, $(\mathrm{Quant}_{(S,\Phi,\Xi)})_K$ is equivalent to R-Mat, and condition (ii) is satisfied as well. Finally, condition (iii) is guaranteed by the hypotheses of the theorem. □

Proof of Corollary A.3. We have a quadratic extension $\mathbb{F}_{p^n}(\sqrt{\varepsilon})/\mathbb{F}_{p^n}$, with 2-element Galois group generated by the involution $\sigma := \xi \mapsto \xi^*$, and corresponding field norm $N_{\mathbb{F}_{p^n}(\sqrt{\varepsilon})/\mathbb{F}_{p^n}}(\xi) := \xi^*\xi$. By Hilbert's Theorem 90, the multiplicative group of those $\xi \in \mathbb{F}_{p^{2n}}$ such that $\xi^*\xi = 1$ is isomorphic to the quotient group $\mathbb{F}_{p^n}(\sqrt{\varepsilon})^\times/\mathbb{F}_{p^n}^\times$. But $\mathbb{F}_{p^n}(\sqrt{\varepsilon})^\times \cong \mathbb{F}_{p^{2n}}^\times$ is cyclic with $p^{2n}-1$ elements, and $\mathbb{F}_{p^n}^\times$ has p^n-1 elements: hence the quotient is cyclic with $(p^{2n}-1)/(p^n-1) = p^n+1$ elements, i.e. it is $\mathbb{Z}_{p^n+1}$. □

Proof of Lemma A.4. Because $\mathbb{F}_{p^n}(\sqrt{\varepsilon})$ is a field, we have that $a^*a = b^*b$ if and only if $a = \xi b$ for some ξ such that $\xi^*\xi = 1$, i.e. for some phase ξ. Equality up to phase is an equivalence relation on elements of $\mathbb{F}_{p^n}(\sqrt{\varepsilon})$ (because phases form a group under multiplication), and there are exactly p^n+1 phases by Corollary A.3: as a consequence, there are exactly $(p^{2n}-1)/(p^n+1) = p^n-1$ non-zero pure scalars in $\mathrm{CP}^*[\mathbb{F}_{p^n}(\sqrt{\varepsilon})\text{-Mat}]$, i.e. all the scalars are in fact pure. □

Proof of Lemma A.5 By Corollary A.3, the phases of $\mathbb{F}_{p^n}(\sqrt{\varepsilon})$ form the finite cyclic group $\mathbb{Z}_{p^n+1}$, and hence the $\mathbb{F}_{p^n}(\sqrt{\varepsilon})$-valued unitary multiplicative characters of G are exactly the group homomorphisms $G \to \mathbb{Z}_{p^n+1}$. The unitary multiplicative characters of a product $\prod_{k=1}^{K} \mathbb{Z}_{p_k^{e_k}}$ (where $p_1, ..., p_K$ are pairwise distinct primes) take the form $(g_1, ..., g_K) \mapsto \chi_1(g_1) \cdot ... \cdot \chi_K(g_K)$, where $(\chi_1, ..., \chi_K)$ are all possible K-tuples where each χ_k is a unitary multiplicative character of the corresponding factor $\mathbb{Z}_{p_k^{e_k}}$. Hence $G \cong \prod_{k=1}^{K} \mathbb{Z}_{p_k^{e_k}}$ has enough multiplicative characters if and only if each factor $\mathbb{Z}_{p_k^{e_k}}$ does, and in turn this is true if and only if $p_k^{e_k} | p^n + 1$ for all $k = 1, ..., K$. The final statement about the Hidden Subgroup Problem is a consequence of the main result from Ref. [34]: because all positive elements are pure scalars, is is always true that $|G| = z_G^* z_G$ for some $z_G \in \mathbb{F}_{p^n}(\sqrt{\varepsilon})$, and furthermore $|G|$ is always invertible because we must necessarily have that p does not divide $|G|$ (otherwise we would get $p | p^n + 1$, which is absurd). $\square$

Proof of Lemma A.6. If $q^2 | p^n + 1$, we can consider the following argument. We take the subgroup of classical states to be $K \cong \mathbb{Z}_q$, seen as the subgroup $K = \langle (\frac{p^n+1}{q}, 2\frac{p^n+1}{q}, ..., (q-1)\frac{p^n+1}{q}) \rangle \lhd \mathbb{Z}_{p^n+1}^{q-1}$, and we use the equation $qy = (\frac{p^n+1}{q}, 2\frac{p^n+1}{q}, ..., (q-1)\frac{p^n+1}{q})$. The equation cannot have any solution in K, where $qy = (0, 0, ..., 0)$ for all y, but has solution $y = (\frac{p^n+1}{q^2}, 2\frac{p^n+1}{q^2}, ..., (q-1)\frac{p^n+1}{q^2})$ in the group of phase gates $\mathbb{Z}_{p^n+1}^{q-1}$. Conversely, if $p^n + 1 = \prod_{k=1}^{K} p_k$ for distinct primes $p_1, ..., p_K$, then for any classical subgroup K the group of phase gates decomposes as $K \times K'$ for some K', and a result of [37] shows that no non-trivial generalised Mermin-type argument can be formulated. We have used the fact that $|\mathbb{Z}_q|$ is always in the form $|\mathbb{Z}_q| = q = z_q^* z_q$ for some $z_q \in \mathbb{F}_{p^n}(\sqrt{\varepsilon})$: this is because all positive elements are pure scalars and $|\mathbb{Z}_q|$ must be invertible (p cannot divide q, otherwise we would get $p | p^n + 1$). $\square$

Proof of Lemma A.9. Let * be an involution for the tropical semiring S: $x \leq y$ implies that $x = x + y$, so that $x^* = x^* + y^*$ and $x^* \leq y^*$. But then $x \leq x^*$ implies $x^* \leq (x^*)^* = x$ (and similarly for $x^* \leq x$), so that $x^* = x$ is the trivial involution. Now consider the tropical semiring with trivial involution, so that the positive elements are exactly those in the form x^2 for some $x \in S$. But in a tropical semiring we have that $x^2 + y^2 = (x + y)^2$ (as Speyer and Sturmfels put it, "the Freshman's dream holds in tropical arithmetic." [70]): hence the squares are closed under addition $+$, and form a sub-semiring. $\square$